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Application of the coherent state formalism to multiply excited states

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Abstract

A general expression is obtained for the matrix element of an m-body operator between coherent states constructed from multiple orthogonal coherent boson species. This allows the coherent state formalism to be applied to states possessing an arbitrarily large number of intrinsic excitation quanta. For illustration, the formalism is applied to the two-dimensional vibron model ($U(3)$ model), to calculate the energies of all excited states in the large-N limit.

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1. Introduction

Methods based upon coherent states [1] have proved to be widely useful in investigating the ground state properties of algebraic models. In algebraic models, the Hamiltonian and all physical operators are constructed from the elements of a Lie algebra, usually from a bosonic realization of $U(n)$. Such models have been applied extensively to the spectroscopy of many-body systems, including nuclei [2] and molecules [3]. The coherent states of an algebraic model are obtained by repeated action of a general linear combination of boson creation operators on the vacuum state. As variational trial states, the coherent states allow the estimation of the ground state energies and properties, yielding results which become exact in the infinite boson number limit [4, 5]. They are also essential in defining the classical limit for the model [5], providing the geometric coordinates or dynamical variables of the model.

Coherent states may also be used to study the intrinsic excitation modes of an algebraic system [6–15], through the construction of coherent states orthogonal to the ground state. However, application of this method has generally been limited to excited states involving only one intrinsic excitation quantum, or at most two [15], due to the complexity of calculating the necessary matrix elements. In the present work, a general expression is obtained for the matrix element of an m-body operator between coherent states constructed from multiple orthogonal linear combinations of boson creation operators. This allows the coherent state formalism to be applied to states possessing an arbitrarily large number of intrinsic excitation quanta.

The necessary definitions for the basic ground state (condensate) coherent state are first presented, and the results for expectation values with respect to this state are reviewed (section 2). The general result for excited coherent states is then established (section 3). As a simple illustration, calculations are carried out for excited states in the $SO(3)$ dynamical symmetry limit of the molecular two-dimensional vibron model [16] (section 4).

2. Condensate coherent state

An algebraic model based upon a bosonic realization of $U(n)$ is obtained by defining n bosonic creation operators $b_1^\dagger, b_2^\dagger, \ldots, b_n^\dagger$ obeying the canonical commutation relations $[b_i, b_j] = 0$, $\left[b_i, b_j^\dagger\right] = \delta_{i,j}$ and $\left[b_i^\dagger, b_j^\dagger\right] = 0$. The set of all possible bilinears of a creation and an annihilation operator then forms a basis $b_1^\dagger b_1, b_1^\dagger b_2, \ldots, b_n^\dagger b_n$ for the algebra. The physical operators of the model, such as the Hamiltonian, the angular momentum and transition operators, are constructed as polynomials in the $b_i^\dagger b_j$. These operators act on the states of the Fock space created by $b_1^\dagger, b_2^\dagger, \ldots, b_n^\dagger$.

The condensate coherent state is defined in terms of the 'condensate boson' creation operator, which is a general linear combination

$$B_c^\dagger \equiv \alpha_1 b_1^\dagger + \alpha_2 b_2^\dagger + \cdots + \alpha_n b_n^\dagger, \tag{1}$$

with complex coefficients satisfying the normalization convention $\sum_{i=1}^n \alpha_i^* \alpha_i = 1$. The normalized condensate coherent state is then

$$|N; \alpha_1, \ldots, \alpha_n\rangle \equiv \frac{1}{\sqrt{N!}} \left(B_c^\dagger\right)^N |0\rangle. \tag{2}$$

This state is an eigenstate of the total number operator $\hat{N} \equiv \sum_{i=1}^n b_i^\dagger b_i$.

The expectation value of a one-body or two-body operator with respect to the condensate (2) was deduced by Van Isacker and Chen [17], using arguments based upon formal differentiation. Here let us derive an explicit result for the expectation value of an arbitrary m-body operator, since the results of section 3 can be obtained as a natural extension. First, for an annihilation operator b_r, note that $\left[b_r, B_c^\dagger\right] = \alpha_r$, from which the relation

$$\left[b_r, \left(B_c^\dagger\right)^N\right] = N\alpha_r \left(B_c^\dagger\right)^{N-1} \tag{3}$$

follows by the product rule for commutators. The action of b_r on the condensate is thus

$$b_r |N; \alpha_1, \ldots, \alpha_n\rangle = \sqrt{N}\alpha_r |N-1; \alpha_1, \ldots, \alpha_n\rangle. \tag{4}$$

Repeated application yields the expectation value of an arbitrary m-body operator,

$$\langle N; \alpha_1, \ldots, \alpha_n| \left(\prod_{i=1}^m b_{r_i'}^\dagger\right) \left(\prod_{i=1}^m b_{r_i}\right) |N; \alpha_1, \ldots, \alpha_n\rangle = N^{\underline{m}} \prod_{i=1}^m \alpha_{r_i'}^* \alpha_{r_i}, \tag{5}$$

where the underlined superscript indicates the falling factorial [18], defined by $m^{\underline{n}} \equiv m(m-1)\cdots(m-n+1)$.

3. General coherent state

For the study of excited states, it is necessary to consider coherent states which are orthogonal to the condensate state. These are constructed using multiple different coherent boson species $B_s^\dagger$ $(s = 1, \ldots, S)$, defined as linear combinations

$$B_s^\dagger \equiv \alpha_{s,1} b_1^\dagger + \alpha_{s,2} b_2^\dagger + \cdots + \alpha_{s,n} b_n^\dagger \tag{6}$$

of the basic boson creation operators. (The procedure for obtaining values of the coefficients $\alpha_{s,i}$ appropriate to a given model is discussed in, e.g., [11].) The coefficients $\alpha_{s,i}$ are chosen to obey the orthonormalization convention $\sum_{i=1}^{n} \alpha^*_{s',i}\alpha_{s,i} = \delta_{s',s}$. Consequently, the coherent bosons satisfy canonical commutation relations $[B_{s'}, B_s] = 0$, $\left[B_{s'}, B_s^\dagger\right] = \delta_{s',s}$ and $\left[B_{s'}^\dagger, B_s^\dagger\right] = 0$. The different $B_s^\dagger$ represent the ground state condensate boson and one or more orthogonal excitation modes. The normalized multi-species coherent state is

$$|N_1 \cdots N_S\rangle \equiv \left(\prod_{s=1}^{S} \frac{1}{\sqrt{N_s!}}\left(B_s^\dagger\right)^{N_s}\right)|0\rangle. \tag{7}$$

The coherent state is an eigenstate of the total number operator, of eigenvalue $N = \sum_{s=1}^{S} N_s$. States of different coherent boson occupation numbers $N_1, \ldots, N_S$ are orthogonal.

The matrix element of an arbitrary m-body operator with respect to two multi-species coherent states can now be deduced following the approach of section 2. The commutation relations generalize to $\left[b_r, B_s^\dagger\right] = \alpha_{s,r}$ and

$$\left[b_r, \prod_{s=1}^{S}\left(B_s^\dagger\right)^{N_s}\right] = \sum_{t=1}^{S} N_t\alpha_{t,r}\left[\prod_{s=1}^{S}\left(B_s^\dagger\right)^{N_s-\delta_{s,t}}\right]. \tag{8}$$

The action of b_r on the multi-species coherent state is thus

$$b_r|N_1 \cdots N_S\rangle = \sum_{t=1}^{S}\sqrt{N_t}\alpha_{t,r}|(N_1 - \delta_{1,t}) \cdots (N_S - \delta_{S,t})\rangle. \tag{9}$$

This is analogous to the result of equation (4), but with a separate term arising from the action of b_r on each species of coherent boson contributing to the coherent state. It is useful to define a counting function $\nu(t_1, \ldots, t_m; s) \equiv \sum_{i=1}^{m} \delta_{t_i,s}$, giving the number of the t_i which are equal to s. Then, for a product of annihilation operators acting on the coherent state,

$$\left(\prod_{i=1}^{m} b_{r_i}\right)|N_1 \cdots N_S\rangle = \sum_{t_1,\ldots,t_m=1}^{S}\left[\prod_{s=1}^{S}\sqrt{N_s^{\underline{\nu(t_1,\ldots,t_m;s)}}}\right]\left(\prod_{i=1}^{m}\alpha_{t_i,r_i}\right)$$
$$\times|(N_1 - \nu(t_1, \ldots, t_m; 1)) \cdots (N_S - \nu(t_1, \ldots, t_m; S))\rangle. \tag{10}$$

The matrix element of an arbitrary m-body operator ($m \geqslant 1$) with respect to two arbitrary multi-species coherent states is the inner product of two such expressions,

$$\langle N_1' \cdots N_S'|\left(\prod_{i=1}^{m} b_{r_i'}^\dagger\right)\left(\prod_{i=1}^{m} b_{r_i}\right)|N_1 \cdots N_S\rangle$$
$$= \sum_{\substack{t_1',\ldots,t_m' \\ t_1,\ldots,t_m}=1}^{S}\left[\prod_{s=1}^{S}\delta_{N_s'-\nu_s',N_s-\nu_s}\sqrt{N_s'^{\underline{\nu_s'}}N_s^{\underline{\nu_s}}}\right]\left(\prod_{i=1}^{m}\alpha^*_{t_i',r_i'}\alpha_{t_i,r_i}\right), \tag{11}$$

where the abbreviations $\nu_s' \equiv \nu(t_1', \ldots, t_m'; s)$ and $\nu_s \equiv \nu(t_1, \ldots, t_m; s)$ have been used.

Three stages are involved in evaluating the matrix element of a general operator: re-expression of the operator in terms of normal-ordered m-body terms, evaluation of the matrix elements of these by equation (11), and simplification of the result. For complicated operators or if many coherent boson species are involved, these steps can most effectively be carried out though computer-based symbolic manipulation. A few useful special cases of equation (11) are summarized in the appendix.

The multiple sum in equation (11) nominally contains S^{2m} terms. However, for fixed numerical values of the N_s and N_s', many of the terms vanish identically due to the restriction

Table 1. The number of terms in general contributing to the sum in equation (11), not vanishing due to the delta symbol constraint, for various values of S and m. In practice, some of these terms may also vanish as a consequence of zero values for the falling factorials or $\alpha_{s,i}$ coefficients.

	Contributing terms			
S	One-body	Two-body	Three-body	Four-body
2	2	6	20	70
3	3	15	93	639
4	4	28	256	2716
5	5	45	545	7885

on indices imposed by the product of Kronecker delta symbols. A summary of the number of nonvanishing terms for various S and m is given in table 1. A given species s of coherent boson is overannihilated when $N_s - \nu(s) < 0$ or $N'_s - \nu'(s) < 0$, yielding a vanishing falling factorial in equation (11). Thus, additional terms vanish if the expression is evaluated for a small value ($<m$) of any of the N_s or N'_s, as typically occurs when the lowest lying excited states are considered. If the multiple sum in equation (11) is instead to be evaluated with the N_s and N'_s retained as variables, all terms involving the same product of falling factorials may be collected. This product is identical for terms with the same values of all the $\nu(s)$ and $\nu'(s)$. Since $0 \leqslant \nu(s) \leqslant m$ and $\sum_{s=1}^{S} \nu(s) = m$, and similarly for $\nu'(s)$, the number of distinct terms after collection is the square of the number of possible partitions of m over S bins.

Multi-species coherent states of the form (7) are also encountered as the condensate states of systems involving multiple constituents, each separately conserved. An example from nuclear physics is the proton–neutron interacting boson model (IBM-2) [2], in which proton pairs (created by $s^\dagger_{\pi,0}$, $d^\dagger_{\pi,-2}$, $d^\dagger_{\pi,-1}$, $d^\dagger_{\pi,0}$, $d^\dagger_{\pi,+1}$, $d^\dagger_{\pi,+2}$) and neutron pairs (created by $s^\dagger_{\nu,0}$, $d^\dagger_{\nu,-2}$, $d^\dagger_{\nu,-1}$, $d^\dagger_{\nu,0}$, $d^\dagger_{\nu,+1}$, $d^\dagger_{\nu,+2}$) are separately conserved. The physical operators are constructed from the elements of a Lie algebra $U_1(n) \otimes U_2(n) \otimes \cdots$, and a condensate state with good boson number for each constituent is constructed as $\propto \left(B^\dagger_{c1}\right)^{N_1}\left(B^\dagger_{c2}\right)^{N_2} \cdots |0\rangle$. Since the condensate bosons $B^\dagger_{c\rho}$ ($\rho = 1, 2, \ldots$) are constructed from disjoint sets of boson operators, the expectation value of an m-body operator in general factorizes into the product of simple expectation values of type (5) (e.g., [19], (C1)). However, the general result (11) for the multi-species coherent state matrix element can provide the simplest framework for computer-based symbolic evaluation [20].

4. Intrinsic excitations of the two-dimensional vibron model

The two-dimensional vibron model [16] is the $U(3)$ algebraic model, describing a system containing a dipole degree of freedom constrained to planar motion. The basic example of such a system is a triatomic linear bender molecule, but the model is easily extended to more complex molecular systems. The $U(3)$ algebra is realized in terms of the three bosonic operators $\sigma^\dagger$, $\tau^\dagger_x$ and $\tau^\dagger_y$, which satisfy canonical commutation relations. It is convenient to define circular bosons $\tau^\dagger_\pm \equiv \mp\left(\tau^\dagger_x \pm \mathrm{i}\tau^\dagger_y\right)/\sqrt{2}$ [21, 22][1], such that the operators $\sigma^\dagger$, $\tau^\dagger_+$ and $\tau^\dagger_-$ carry $0, +1$ and -1 units of two-dimensional angular momentum. The physical

[1] The definitions $\tau^\dagger_\pm = \mp(\tau^\dagger_x \pm \mathrm{i}\tau^\dagger_y)/\sqrt{2}$ [21, 22] are used, rather than $\tau^\dagger_\pm = (\tau^\dagger_x \pm \mathrm{i}\tau^\dagger_y)/\sqrt{2}$ [16]. This choice is necessary for $\hat{D}_\pm$ to be the physical dipole operators and provides a closer correspondence with the treatment of the three-dimensional vibron model [11].

operators include the angular momentum operator $\hat{l} \equiv \tau_+^\dagger \tau_+ - \tau_-^\dagger \tau_-$, the dipole operators $\hat{D}_\pm \equiv \pm\sqrt{2}\left(\tau_\pm^\dagger \sigma - \sigma^\dagger \tau_\mp\right)$, and the quadrupole operators $\hat{Q}_\pm \equiv \sqrt{2}\tau_\pm^\dagger \tau_\mp$.

The $U(3)$ algebra contains the subalgebra chains [16]

$$U(3) \supset \begin{cases} U(2) \supset SO(2) \\ SO(3) \supset SO(2). \end{cases} \tag{12}$$

The dynamical symmetry associated with the $U(2)$ chain yields spectra matching those of the cylindrical oscillator (the Pöschl–Teller potential), while the dynamical symmetry associated with the $SO(3)$ chain yields spectra like those of the displaced cylindrical oscillator (the Morse potential). The $SO(3)$ limit is used in the following illustrations, as the less trivial case. The $SO(3)$ subalgebra is spanned by $\{\hat{D}_+, \hat{D}_-, \hat{l}\}$ and has the quadratic Casimir operator $\hat{W}^2 \equiv (\hat{D}_+\hat{D}_- + \hat{D}_-\hat{D}_+)/2 + \hat{l}^2$. The $SO(2)$ subalgebra is simply the two-dimensional angular momentum algebra, containing $\hat{l}$. The simplest Hamiltonian with $SO(3)$ dynamical symmetry is $H = -\hat{W}$, which has eigenvalues

$$E(N, v, l) = -N(N+1) + 4v[(N + 1/2) - v], \tag{13}$$

with $v = 0, 1, \ldots, \lfloor N/2 \rfloor$ and $l = -(N-2v), -(N-2v)+1, \ldots, +(N-2v)$ (figure 1(*a*)).

The condensate boson and an orthogonal excitation boson for the two-dimensional vibron model may be defined as

$$B_c^\dagger(r) \equiv \frac{1}{\sqrt{1+r^2}}\left(\sigma^\dagger + r\tau_x^\dagger\right) \qquad B_x^\dagger(r) \equiv \frac{1}{\sqrt{1+r^2}}\left(-r\sigma^\dagger + \tau_x^\dagger\right). \tag{14}$$

(See [11] for further discussion of the choice of boson operators in the vibron model.) The general excited coherent state is

$$|NN_x; r\rangle \equiv \frac{1}{\sqrt{(N-N_x)!N_x!}}\left[B_c^\dagger(r)\right]^{N-N_x}\left[B_x^\dagger(r)\right]^{N_x}|0\rangle. \tag{15}$$

The expectation value of the $SO(3)$ Casimir operator with respect to an arbitrary excited coherent state is evaluated using equation (A.2), yielding

$$\langle NN_x; r|\hat{W}^2|NN_x; r\rangle = 2[N + N_x(N-N_x)] + \frac{4}{(1+r^2)^2}[N(N-1) - 6N_x(N-N_x)]r^2. \tag{16}$$

The equilibrium value of r is found by minimization of the variational energy $\langle N0; r|H|N0; r\rangle$, giving $r = 1$. With this value of r, the excited coherent state energies are

$$E(N, N_x) = -N(N+1) + 4N_x[N - N_x]. \tag{17}$$

If the intrinsic excitation number N_x is identified with the $SO(3)$ quantum number v, comparison of equations (13) and (17) shows that these expressions differ only by a term of order $1/N$. Thus, to leading order in $1/N$, the coherent state estimate reproduces the excitation energies for all excited states, as illustrated in figure 1(*b*).

Transition strengths may be estimated using coherent states, from the squared matrix element of the transition operator, $I \approx \langle NN_x'|\hat{T}|NN_x\rangle^2$. The coherent state $|NN_x\rangle$ is not an angular momentum eigenstate, so the resulting estimate for the transition strength between two intrinsic excitations is effectively averaged over the many states of different angular momenta constituting that excitation. This coherent state estimate thus cannot be expected to provide the exact transition intensity between any two particular angular momentum eigenstates. It does, however, indicate the general magnitude of the transition strengths and the overall dependence on the excitation quantum number, and it can be quantitatively accurate if the angular momentum dependence of transition strengths is weak. (Alternatively, angular

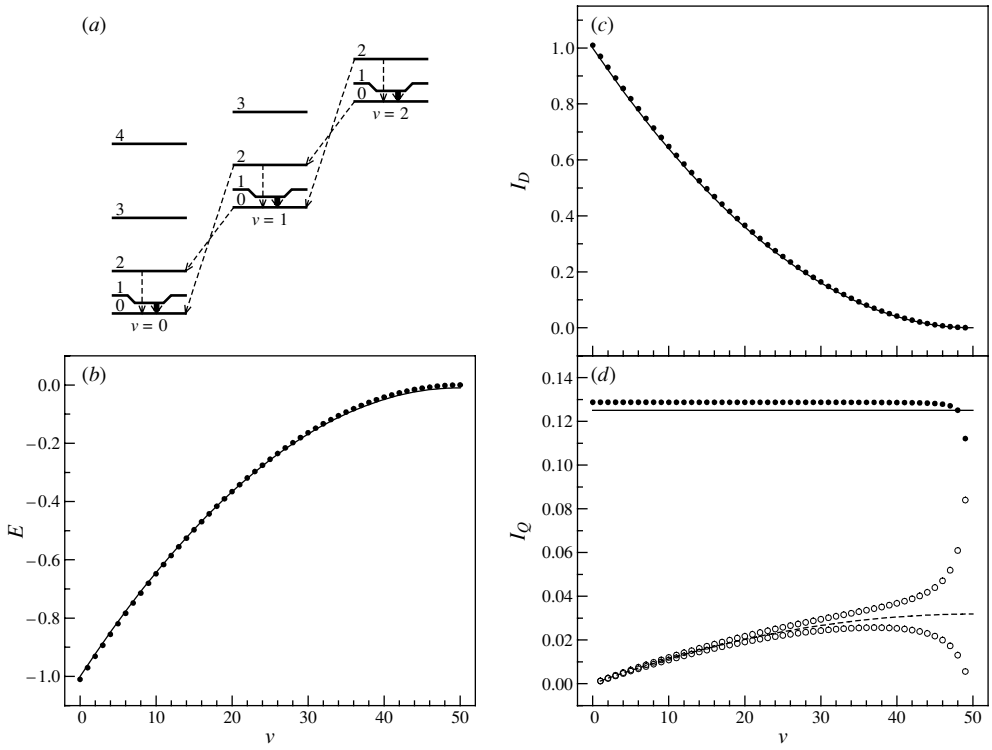

Figure 1. Comparison of excited coherent state estimates of observables (curves) with exact values (circles) for the $SO(3)$ dynamical symmetry of the two-dimensional vibron model, with $N = 100$. (*a*) A schematic $SO(3)$ level energy diagram, with arrows indicating the dipole (solid) and quadrupole (dashed) transitions considered in panels (*c*) and (*d*). (*b*) Energies of all $SO(3)$ representations. (*c*) Intensity of the $(l = 1) \rightarrow (l = 0)$ dipole transition within each representation. (*d*) Intensity of the $(l = 2) \rightarrow (l = 0)$ quadrupole transition within each representation (the solid curve, full circles) and of the $(l = 2) \rightarrow (l = 0)$ and $(l = 0) \rightarrow (l = 2)$ $\Delta v = -1$ transitions between representations (the dashed curve, open circles). An l^2 energy splitting is included in panel (*a*) to provide visual separation of levels within an $SO(3)$ representation. All observables are plotted rescaled by $1/N^2$.

momentum eigenstates may be projected from the coherent states [9, 11, 23], but this requires additional machinery beyond simple evaluation of an m-body operator matrix element.)

Dipole infrared transitions in the two-dimensional vibron model are, to leading order, induced by the operators $\hat{D}_\pm$, and quadrupole Raman transitions are induced by the operators $\hat{Q}_\pm$ [3, 16]. The strengths of transitions within an intrinsic excitation are estimated from the expectation values

$$\begin{aligned} \langle NN_x; r|\hat{D}_\pm|NN_x; r\rangle &= -2(N-2N_x)\frac{r}{1+r^2} \\ \langle NN_x; r|\hat{Q}_\pm|NN_x; r\rangle &= -\frac{1}{\sqrt{2}}\frac{N_x+(N-N_x)r^2}{1+r^2}, \end{aligned} \tag{18}$$

and those between successive intrinsic excitations from

$$\begin{aligned} \langle N(N_x-1); r|\hat{D}_\pm|NN_x; r\rangle &= -\sqrt{(N-N_x+1)N_x}\frac{1-r^2}{1+r^2} \\ \langle N(N_x-1); r|\hat{Q}_\pm|NN_x; r\rangle &= -\frac{1}{\sqrt{2}}\sqrt{(N-N_x+1)N_x}\frac{r}{1+r^2}. \end{aligned} \tag{19}$$

The coherent state estimates of transition intensities for the $SO(3)$ dynamical symmetry are obtained from these equations with $r = 1$. The estimates are plotted for $N = 100$ in figures 1(c) and (d), together with exact values obtained by numerical diagonalization, as functions of the excitation quantum number v (or N_x). The coherent state estimate for dipole transitions closely reproduces the strength of the angular momentum $1 \rightarrow 0$ transition within an $SO(3)$ representation (figure 1(c)). Dipole transitions between different representations are forbidden, and the coherent state estimate indeed vanishes. Quadrupole transition strengths exhibit greater angular momentum dependence within a representation, and the coherent state estimate is consequently less accurate. The strengths of quadrupole transitions involving the low angular momentum members of the representations are reproduced to within $\sim$5%, except at the highest intrinsic excitation quantum numbers (figure 1(d)). Note that the angular momentum $2 \rightarrow 0$ and $0 \rightarrow 2$ transitions between two representations ($\Delta v = \pm 1$) differ in strength, and the coherent state estimate consistently behaves as their average.

5. Conclusion

The present results serve as a basis for application of the coherent state formalism to states with an arbitrary number of intrinsic excitation quanta. This process yields estimates of eigenvalues and operator matrix elements for excited states valid to leading order in $1/N$. The illustration provided was to a dynamical symmetry limit of a simple model, but the coherent state analysis will likely be most useful when applied to transitional Hamiltonians, for which analytic results are not otherwise available.

The coherent state formalism has in the past provided not only a quantitative calculational tool but, perhaps more importantly, a method for obtaining qualitative understanding of the equilibrium properties and fundamental modes of a system. Most, if not all, of the raw numerical results of the coherent state formalism can also be obtained by numerical diagonalization. It is thus the latter, interpretational aspects of the coherent state formalism, and the explicit analytic forms obtained for the parameter dependences of observables, which have proved most useful. The present results for multiply excited states thus might most productively be used in investigating the general nature of the evolution of a system's properties with excitation energy.

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Appendix. Special cases of the general matrix element

Some useful special cases of the general matrix element (11) are given here. If the coherent states involve only two species of coherent boson ($S = 2$), then the one-body and two-body operator matrix elements are

$$\begin{aligned}
&\langle (N_1 - 1)(N_2 + 1)|b^\dagger_{r'} b_r|N_1 N_2\rangle = \sqrt{N_1(N_2 + 1)}\alpha^*_{2,r'}\alpha_{1,r} \\
&\langle N_1 N_2|b^\dagger_{r'} b_r|N_1 N_2\rangle = N_1\alpha^*_{1,r'}\alpha_{1,r} + N_2\alpha^*_{2,r'}\alpha_{2,r} \\
&\langle (N_1 + 1)(N_2 - 1)|b^\dagger_{r'} b_r|N_1 N_2\rangle = \sqrt{(N_1 + 1)N_2}\alpha^*_{1,r'}\alpha_{2,r}
\end{aligned} \tag{A.1}$$

and

$$\langle (N_1-2)(N_2+2)|b^\dagger_{r_2'}b^\dagger_{r_1'}b_{r_1}b_{r_2}|N_1N_2\rangle = \sqrt{N_1^{\underline{2}}(N_2+2)^{\underline{2}}}\alpha^*_{2,r_2'}\alpha^*_{2,r_1'}\alpha_{1,r_1}\alpha_{1,r_2}$$

$$\begin{aligned}\langle (N_1-1)(N_2+1)|b^\dagger_{r_2'}b^\dagger_{r_1'}b_{r_1}b_{r_2}|N_1N_2\rangle &= (N_1-1)\sqrt{N_1(N_2+1)}\big(\alpha^*_{1,r_2'}\alpha^*_{2,r_1'}+\alpha^*_{2,r_2'}\alpha^*_{1,r_1'}\big)\alpha_{1,r_1}\alpha_{1,r_2}\\ &\quad + N_2\sqrt{N_1(N_2+1)}\alpha^*_{2,r_2'}\alpha^*_{2,r_1'}\big(\alpha_{2,r_1}\alpha_{1,r_2}+\alpha_{1,r_1}\alpha_{2,r_2}\big)\end{aligned}$$

$$\begin{aligned}\langle N_1N_2|b^\dagger_{r_2'}b^\dagger_{r_1'}b_{r_1}b_{r_2}|N_1N_2\rangle &= N_1^{\underline{2}}\alpha^*_{1,r_2'}\alpha^*_{1,r_1'}\alpha_{1,r_1}\alpha_{1,r_2}+N_2^{\underline{2}}\alpha^*_{2,r_2'}\alpha^*_{2,r_1'}\alpha_{2,r_1}\alpha_{2,r_2}\\ &\quad + N_1N_2\big(\alpha^*_{1,r_2'}\alpha^*_{2,r_1'}+\alpha^*_{2,r_2'}\alpha^*_{1,r_1'}\big)\big(\alpha_{1,r_1}\alpha_{2,r_2}+\alpha_{2,r_1}\alpha_{1,r_2}\big)\end{aligned} \tag{A.2}$$

$$\begin{aligned}\langle (N_1+1)(N_2-1)|b^\dagger_{r_2'}b^\dagger_{r_1'}b_{r_1}b_{r_2}|N_1N_2\rangle &= N_1\sqrt{(N_1+1)N_2}\alpha^*_{1,r_2'}\alpha^*_{1,r_1'}\big(\alpha_{2,r_1}\alpha_{1,r_2}+\alpha_{1,r_1}\alpha_{2,r_2}\big)\\ &\quad + (N_2-1)\sqrt{(N_1+1)N_2}\big(\alpha^*_{1,r_2'}\alpha^*_{2,r_1'}+\alpha^*_{2,r_2'}\alpha^*_{1,r_1'}\big)\alpha_{2,r_1}\alpha_{2,r_2}\end{aligned}$$

$$\langle (N_1+2)(N_2-2)|b^\dagger_{r_2'}b^\dagger_{r_1'}b_{r_1}b_{r_2}|N_1N_2\rangle = \sqrt{(N_1+2)^{\underline{2}}N_2^{\underline{2}}}\alpha^*_{1,r_2'}\alpha^*_{1,r_1'}\alpha_{2,r_1}\alpha_{2,r_2},$$

with all others zero. For an expectation value (all $N_s' = N_s$), equation (11) simplifies to

$$\langle N_1\cdots N_S|\left(\prod_{i=1}^m b^\dagger_{r_i'}\right)\left(\prod_{i=1}^m b_{r_i}\right)|N_1\cdots N_S\rangle = \sum_{\substack{t_1',\ldots,t_m'\\ t_1,\ldots,t_m}=1}^{S}\left[\prod_{s=1}^S \delta_{\nu_s',\nu_s}N_s^{\underline{\nu_s}}\right]\left(\prod_{i=1}^m \alpha^*_{t_i',r_i'}\alpha_{t_i,r_i}\right). \tag{A.3}$$

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